Resit Exam — Functional Analysis (WIFA–08)

Tuesday 27 June 2017, 9.00h–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (7 + 10 + 8 = 25 points)

Let X be a finite-dimensional linear space over a field $\mathbb{K}$. Write $X = \text{span}\,\{e_1, \dots, e_d\}$ and define

$$\|x\|_+ = \max\{|\lambda_i| \,:\, i = 1, \dots, d\} \quad \text{where} \quad x = \lambda_1 e_1 + \dots + \lambda_d e_d, \quad \lambda_i \in \mathbb{K}.$$

Prove the following statements:

(a) $\|\cdot\|_+$ is a norm on X;

(b) $(X, \|\cdot\|_+)$ is a Banach space (i.e., every Cauchy sequence has a limit);

(c) $(X, \|\cdot\|)$ is a Banach space for any other norm $\|\cdot\|$ on X.

Problem 2 (10 + 8 + 7 = 25 points)

Let $X = \mathcal{C}([a, b], \mathbb{K})$ be provided with the supremum norm. Consider the following linear operator:

$$T : X \to X, \quad Tf(x) = xf(x).$$

Prove the following statements:

(a) $\|T\| = \max\{|a|, |b|\}$;

(b) T has no eigenvalues;

(c) $\rho(T) = \mathbb{K} \setminus [a, b]$.

Problem 3 (4 + 3 + 10 + 3 = 20 points)

(a) Formulate the closed graph theorem.

(b) Define the linear subspace $V \subset \ell^2$ by

$$V = \{(x_1, x_2, x_3, \dots) \in \ell^2 : (x_1, 2x_2, 3x_3, \dots) \in \ell^2\}$$

and consider the linear operator

$$T : V \subset \ell^2 \to \ell^2, \quad (x_1, x_2, x_3, \dots) \mapsto (x_1, 2x_2, 3x_3, \dots).$$

Prove the following statements:

(i) T is *not* bounded;

(ii) T is closed.

(iii) V is *not* closed in ℓ^2.

Problem 4 (5 + 5 + 5 + 5 = 20 points)

Let X be a normed linear space. For nonempty subsets $V \subset X$ and $Z \subset X'$ define

$$V^\perp = \{f \in X' : f(x) = 0 \text{ for all } x \in V\},$$

$$^\perp Z = \{x \in X : f(x) = 0 \text{ for all } f \in Z\}.$$

Prove the following statements:

(a) $V^\perp$ is a linear subspace of X';

(b) $V^\perp$ is closed in X';

(c) $V_1 \subset V_2 \subset X \Rightarrow V_2^\perp \subset V_1^\perp$;

(d) $V \subset {}^\perp(V^\perp)$.

End of test (90 points)

Solution of Problem 1 (7 + 10 + 8 = 25 points)

(a) Clearly, $\|x\|_+ \geq 0$ for all $x \in X$. If $\|x\|_+ = 0$, then $\lambda_i = 0$ for all $i = 1, \dots, d$, which implies that $x = 0$.

(1 point)

The homogeneity of the norm is proven as follows:

$$\begin{aligned} \mu x &= (\mu\lambda_1)e_1 + \cdots + (\mu\lambda_d)e_d \\ \|\mu x\|_+ &= \max\{|\mu\lambda_i| \,:\, i = 1, \dots, d\} \\ &= \max\{|\mu|\,|\lambda_i| \,:\, i = 1, \dots, d\} \\ &= |\mu| \max\{|\lambda_i| \,:\, i = 1, \dots, d\} \\ &= |\mu| \|x\|_+. \end{aligned}$$

(3 points)

Finally, the triangle inequality follows from:

$$\begin{aligned} x + y &= (\lambda_1 + \mu_1)e_1 + \cdots + (\lambda_d + \mu_d)e_d \\ \|x + y\|_+ &= \max\{|\lambda_i + \mu_i| \,:\, i = 1, \dots, d\} \\ &\leq \max\{|\lambda_i| + |\mu_i| \,:\, i = 1, \dots, d\} \\ &\leq \max\{|\lambda_i| \,:\, i = 1, \dots, d\} + \max\{|\mu_i| \,:\, i = 1, \dots, d\} \\ &= \|x\|_+ + \|y\|_+. \end{aligned}$$

(3 points)

(b) If $x_n = \lambda_1^n e_1 + \cdots + \lambda_d^n e_d$ is a Cauchy sequence in $(X, \|\cdot\|_+)$, then for each $\varepsilon > 0$ there exists $N > 0$ such that

$$\begin{aligned} n, m \geq N \quad &\Rightarrow \quad \|x_n - x_m\|_+ \leq \varepsilon \\ &\Rightarrow \quad |\lambda_i^n - \lambda_i^m| \leq \varepsilon \quad \text{for all} \quad i = 1, \dots, d. \end{aligned}$$

This means that (λ_i^n) is a Cauchy sequence in $\mathbb{K}$ for each $i = 1, \dots, n$.

(4 points)

Since $\mathbb{K}$ is complete, $\lambda_i^n \to \lambda_i$ for some $\lambda_i \in \mathbb{K}$.

(2 points)

Let $N_i > 0$ be such that

$$n \geq N_i \quad \Rightarrow \quad |\lambda_i^n - \lambda_i| \leq \varepsilon,$$

and set $M = \max\{N_1, \dots, N_d\}$. Define $x = \lambda_1 e_1 + \cdots + \lambda_d e_d$. Then clearly $x \in X$ and

$$n \geq M \quad \Rightarrow \quad \|x_n - x\|_+ = \max\{|\lambda_i^n - \lambda_i| \,:\, i = 1, \dots, d\} \leq \varepsilon.$$

This shows that $x_n \to x$ in $(X, \|\cdot\|_+)$.

(4 points)

(c) On a finite-dimensional space all norms are equivalent. Hence, there exist constants $a, b > 0$ such that

$$a\|x\|_+ \leq \|x\| \leq b\|x\|_+$$

for all $x \in X$.

(3 points)

The first inequality implies that if (x_n) is a Cauchy sequence with respect to $\|\cdot\|$ then it is also a Cauchy sequence with respect to $\|\cdot\|_+$. By part (b) it follows that there exists $x \in X$ such that $\|x_n - x\|_+ \to 0$.

(3 points)

The second inequality now implies that $\|x_n - x\| \to 0$ as well.

(2 points)

Solution of Problem 2 (10 + 8 + 7 = 25 points)

(a) For all $x \in [a, b]$ we have the following inequality:

$$|Tf(x)| = |xf(x)| = |x|\,|f(x)| \le |x|\,\|f\|_\infty \le \max\{|a|, |b|\}\|f\|_\infty.$$

(4 points)

This implies that

$$\|Tf\|_\infty = \sup_{x\in[a,b]} |Tf(x)| \le \max\{|a|, |b|\}\|f\|_\infty$$

so that

$$\|T\| = \sup_{f\neq 0} \frac{\|Tf\|_\infty}{\|f\|_\infty} \le \max\{|a|, |b|\}.$$

(2 points)

On the other hand, if $f(x) = 1$ for all $x \in [a, b]$, then the last inequality sign becomes an equality.

(4 points)

(b) If $Tf = \lambda f$ then $(x - \lambda)f(x) = 0$ for all $x \in [a, b]$ so that $f(x) = 0$ for all $x \in [a, b] \setminus \{\lambda\}$.

(3 points)

Since f is a continuous function it follows that that $f(x) = 0$ for all $x \in [a, b]$.

(3 points)

We conclude that $Tf = \lambda f$ implies $f = 0$ which means that T has no eigenvalues.

(2 points)

(c) If $\lambda \notin [a, b]$ then $\delta = d(\lambda, [a, b]) > 0$.

(2 points)

Note that for all $x \in [a, b]$ we have

$$|(T - \lambda)^{-1}f(x)| = \left|\frac{f(x)}{x - \lambda}\right| \le \frac{1}{\delta}|f(x)| \le \frac{1}{\delta}\|f\|_\infty$$

which implies that

$$\|(T - \lambda)^{-1}f\|_\infty \le \frac{1}{\delta}\|f\|_\infty$$

(3 points)

Since $(T - \lambda)^{-1}$ is bounded it follows that $\lambda \in \rho(T)$.

(2 points)

Solution of Problem 3 (4 + 3 + 10 + 3 = 20 points)

(a) Assume that X and Y are Banach spaces, $V \subset X$ is a closed linear subspace, and $T : V \to Y$ is a linear operator. If the graph $G(T)$ of T is closed then $T \in B(V, Y)$.

(4 points)

(b) (i) Let $e_n = (0, \dots, 0, 1, 0, \dots)$ where the 1 is at the n-th entry of the sequence. Clearly, $e_n \in V$, $\|e_n\| = 1$, and $Te_n = ne_n$. This implies that $\|Te_n\| = n$ so that

$$\sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \infty,$$

which implies that T is unbounded.

(3 points)

(ii) Let $(x, y) \in \overline{G(T)}$, then there exists a sequence $x^k \in V$ such that $x^k \to x$ and $Tx^k \to y$.

For all $k, n \in \mathbb{N}$ we have

$$|x_n^k - y_n/n| \leq |nx_n^k - y_n| \leq \|Tx^k - y\| \to 0 \quad \text{as} \quad k \to \infty,$$

and

$$|x_n^k - x_n| \leq \|x^k - x\| \to 0 \quad \text{as} \quad k \to \infty.$$

Hence,

$$\frac{y_n}{n} = \lim_{n \to \infty} x_n^k = x_n \quad \text{for all} \quad n\mathbb{N},$$

which implies that $y_n = nx_n$ so that $y = Tx$.

(5 points)

In order to show that $x \in V$ we must prove that $y = Tx \in \ell^2$. Note that

$$\|y\| \leq \|y - Tx^k\| + \|Tx^k\|.$$

The first term on the right hand side goes to zero, whereas the second term on the right hand side is bounded. Hence $\|y\| < \infty$, which means that $y \in \ell^2$. This proves that $(x, y) \in G(T)$ which implies that $G(T)$ is closed.

(5 points)

(iii) If V is closed and T is closed, then by the closed graph theorem it follows that $T \in B(V, Y)$. However, by part (i) it follows that T is unbounded and by part (ii) it follows that T is closed. Hence, V cannot be closed.

(3 points)

Solution of Problem 4 (5 + 5 + 5 + 5 = 20 points)

(a) If $f, g \in V^\perp$ and $\lambda, \mu \in \mathbb{K}$, then $x \in V$ implies that

$$(\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x) = 0,$$

which shows that $\lambda f + \mu g \in V^\perp$ as well.
(5 points)

(b) If $f \in \overline{V^\perp}$, then there exists a sequence $f_n \in V^\perp$ such that $f_n \to f$. Let $x \in V$, then

$$|f(x)| = |f(x) - f_n(x)| \leq \|f_n - f\| \, \|x\| \to 0,$$

which shows that $f \in V^\perp$ as well. We conclude that $V^\perp$ is closed in X'.
(5 points)

(c) If $f \in V_2^\perp$ then $f(x) = 0$ for all $x \in V_2$. Since $V_1 \subset V_2$ it follows that $f(x) = 0$ for all $x \in V_1$ as well. This means that $f \in V_1^\perp$.
(5 points)

(d) If $x \in V$ then $f(x) = 0$ for all $f \in V^\perp$, which in turn implies that $x \in {}^\perp(V^\perp)$.
(5 points)